From epidemic models to the R number

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Outline

- Exponential growth
- A simple epidemic model
- The R number (that everybody has been talking about)
- Connect to real-life work

You might have seen this

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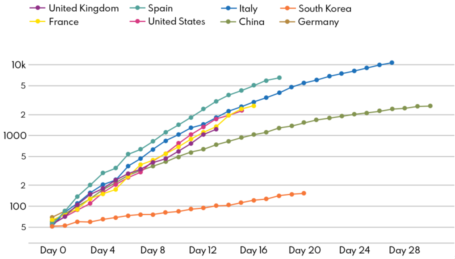
STAY HOME > PROTECT THE NHS > SAVE LIVES

COBR
Cabinet Office Briefing Rooms

Global comparison of deaths

Global deaths comparison.
Countries are aligned by stage of the outbreak. Day 0 equals the first day 50 deaths were reported. (Confidence: deaths are reasonably accurate, but international reporting lags are unclear, so may not be comparing exactly like for like).

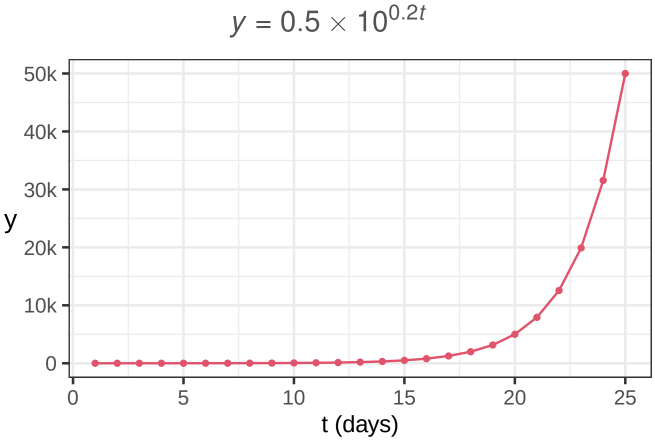


Source: Public Health England, Worldometer. Reporting of UK deaths may lag by up to several days. Logarithmic scale – see international tab for linear scale.

HM Government

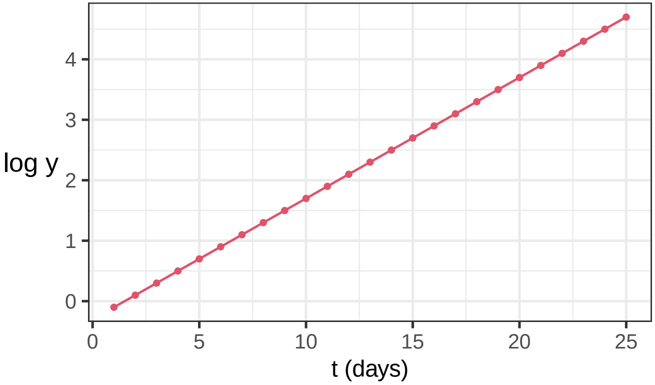
Source: Government's slides to accompany coronavirus press conference

Exponential growth . . .



. . . on a straight line

$$\log_{10} y = \log_{10} 0.5 + 0.2t$$



10-fold every 5 days

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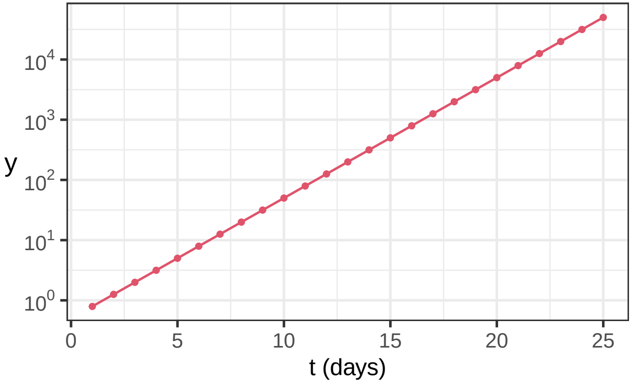
$$\log_{10} y = \log_{10} 0.5 + 0.2t$$



An epidemic model

- Realistically, the number of cases/deaths can't go to infinity
- Susceptible-Infectious (SI) model
  - S: Number of people who have not been infected (susceptible)
  - I: Number of people who are infected and infectious
  - N: Total population, constant over time

$$S + I = N$$

Dynamics of the SI model

- Susceptible or infectious at any time, but not both
- Transition from S to I
  - No contact, no infection
  - Contact with an infectious person - might or might not become infectious
- How do we describe the dynamics? Differential equations!

$$\frac{dS}{dt} = -\beta \frac{S \times I}{N}$$

$$\frac{dI}{dt} = \beta \frac{S \times I}{N}$$

Understanding the equations

- $\beta > 0$ is an unknown **parameter**
- $\frac{dS}{dt} + \frac{dI}{dt} = 0$
- Also, remember $S + I = N$, so

$$\frac{dI}{dt} = \beta \frac{S \times I}{N}$$
$$= \beta \frac{(N - I) \times I}{N}$$

Mock question!

A-Level
MATHEMATICS
Paper 1

Question 1

a. Decompose the function $\frac{N}{(N-I)\times I}$ into partial fractions. [2 marks]

b. Given that $I = 1$ at $t = 0$, solve the following differential equation. [5 marks]

$$\frac{dI}{dt} = \beta\frac{(N-I)\times I}{N}$$

Question 1a

Let

$$\frac{N}{(N-I)\times I} = \frac{A}{N-I} + \frac{B}{I}$$

Rearranging terms, we have

$$\frac{N}{(N-I)\times I} = \frac{(A-B)\times I + BN}{(N-I)\times I}$$

Equating the coefficients, we have $A - B = 0$ and $B = 1$, which means $A = 1$.

$$\therefore \frac{N}{(N-I)\times I} = \frac{1}{N-I} + \frac{1}{I}$$

Question 1b

As this is a "variables separable" question, we have

$$\frac{N}{(N-I)\times I}\frac{dI}{dt}=\beta$$

Integrating both sides with respect to t

$$\int\frac{N}{(N-I)\times I}dI=\int\beta dt$$

Using result in Question 1a,

$$\int\left(\frac{1}{N-I}+\frac{1}{I}\right)dI=\beta t+c,$$

where c is a constant.

Question 1b (cont'd)

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$$-\log_e(N - I) + \log_e I = \beta t + c$$

As $\log_e \frac{A}{B} = \log_e A - \log_e B = -\log_e B + \log_e A$,

$$\log_e \frac{I}{N - I} = \beta t + c$$

Substitute the initial condition $I = 1$ when $t = 0$,

$$\log_e \frac{1}{N - 1} = c$$

Question 1b (cont'd)

$$\log_e \frac{I}{N-I} = \beta t + \log_e \frac{1}{N-1}$$

Exponentiating and reciprocating both sides,

$$\frac{N-I}{I} = (N-1)e^{-\beta t}$$

Adding one to both sides and rearranging,

$$I = \frac{N}{1+(N-1)e^{-\beta t}}$$

Intepretation

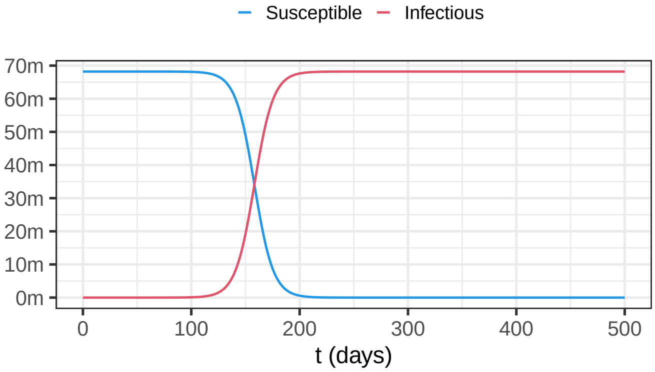
$$I = \frac{N}{1 + (N - 1)e^{-\beta t}}$$

As t becomes larger:

- $e^{-\beta t}$ becomes smaller
- The denominator becomes smaller
- I becomes larger

An example

N = 68.2 million, β = 0.114



A better model

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SIR model

- R: People who are **removed** (recovered / death / vaccinated)
  - Different to the **R number** the government has been talking about
  - We will come to this R number later
- S, I, N: as before

$$S + I + R = N$$

Differential equations again

$$\frac{dS}{dt} = -\beta \frac{S \times I}{N}$$
$$\frac{dI}{dt} = \beta \frac{S \times I}{N} - \gamma I$$
$$\frac{dR}{dt} = \gamma I$$

Observe that

$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = \frac{dN}{dt} = 0$$

Another mock question?

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- Too difficult to solve it in 5 minutes
  - You'll learn how to solve these equations
- In some more sophisticated models
  - Impossible to solve the equations using pen and paper
  - You'll learn numerical methods if a "nice" solution is not available
- Want to know more? See the SI, SIR & other models on Wikipedia

The R_0 number

- Two parameters, β and γ in the SIR model
  - β: The rate of infection
  - γ: The rate of removal (recovery or death)
- The R_0 number is equal to β/γ
- If $\beta < \gamma$, $R_0 < 1$
  - Removals faster than new infections
  - Epidemic under control
- If $\beta > \gamma$, $R_0 > 1$
  - New infections faster than removals
  - Epidemic bound to happen

Another way of looking at R_0

$$\frac{dI}{dt} = \beta \frac{S \times I}{N} - \gamma I$$
$$= \frac{\beta}{\gamma} \times \frac{S \times \gamma I}{N} - \gamma I$$
$$= \left(R_0 \frac{S}{N} - 1 \right) \times \gamma I$$

At $t = 0$, S is close to N, so

$$\frac{dI}{dt} \approx (R_0 - 1) \times \gamma I$$

If $R_0 > 1$, $\frac{dI}{dt} > 0 \Rightarrow$ increasing number of infectious

R_0 changes over time and space

- It seems like the R_0 number is constant
- But β (infection rate) and R_0 depend on some factors:
  - The biological nature of the virus
  - How many contacts do we make
- Social distancing and other measures:
  - lower the 2nd factor
  - push β & R_0 down

More deadly = bigger pandemic?

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- Ebola has a higher death rate
  - γ is higher
- $R_0 = \beta/\gamma$
  - If β stays the same
  - A higher γ pushes R_0 down
- Deadlier diseases/viruses not necessarily more widespread
- One single number doesn't tell the whole story
- Source: R_0 number on Wikipedia

Values of R_0 of well-known infectious diseases

Disease	Transmission	R_0
Measles	Aerosol	12–18[1]
Chickenpox (varicella)	Aerosol	10–12[2]
Mumps	Respiratory droplets	10–12[3]
Rubella	Respiratory droplets	6–7[4]
Polio	Fecal–oral route	5–7[5]
Pertussis	Respiratory droplets	5.5[6]
Smallpox	Respiratory droplets	3.5–6.0[7]
COVID-19 (wild type)	Respiratory droplets and aerosol[8]	2.9 (2.4–3.4)[9]
HIV/AIDS	Body fluids	2–5[10]
SARS	Respiratory droplets	2–4[11]
Common cold	Respiratory droplets	2–3[12]
Diphtheria	Saliva	2.6 (1.7–4.3)[13]
Ebola (2014 Ebola outbreak)	Body fluids	1.78[14]
Influenza (2009 pandemic strain)	Respiratory droplets	1.6 (1.3–2.0)[15]
Influenza (seasonal strains)	Respiratory droplets	1.3 (1.2–1.4)[16]
Nipah virus	Body fluids	0.48[17]
MERS	Respiratory droplets	0.47 (0.29–0.80)[18]

What are the scientists doing?

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- Those performing clinical trials and developing vaccines
- One is no longer susceptible (S) or infected (I) once vaccinated
- Growth of I greatly reduced

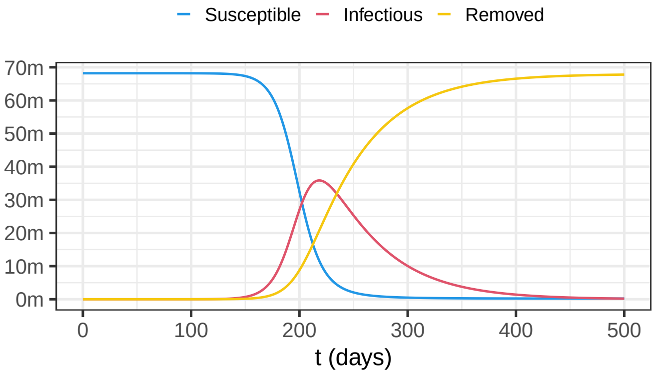
$$\frac{dI}{dt} = \beta \frac{S \times I}{N} - \gamma I$$

What are other scientists (= we) doing?

- Calculate the parameters and R_0 number
- Collect the data of the numbers of infectious, deaths and recovery
- Apply more realistic models
  - The SI and SIR models are too simplistic
  - If the model is no good, the results are no good

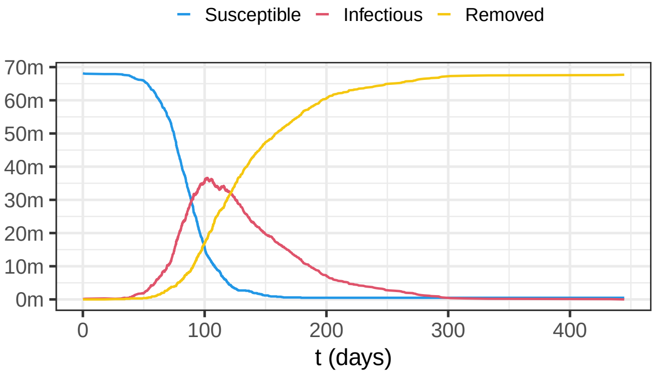
If we know the numbers

- N = 68.2 million
- $\beta = 0.114$, $\gamma = 0.02 \Rightarrow R_0 = 5.7$

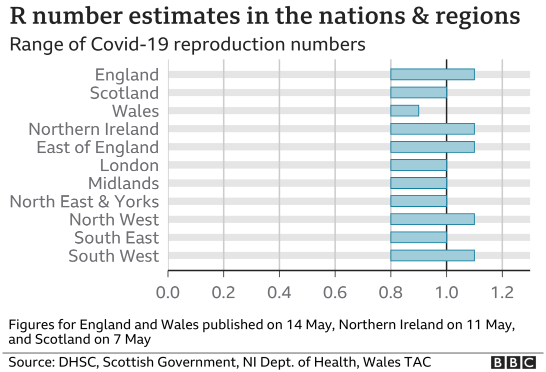


Reality

- Population varying & divided into regions
- β = ______, γ = _____ $\Rightarrow R_0$ = ___



Uncertainty



Source: BBC's coronavirus UK map

Thanks for listening!

Useful resources:

- Government's press conference slides
- Government's coronavirus dashboard